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MODULATION AND PARSEVAL'S THEOREM FOR DISTRIBUTIONAL TWO DIMENSIONAL FOURIER-MELLIN TRANSFORM

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ABSTRACT

The Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform and the Fourier transform is a reversible, linear transform with many important properties. Mathematically, Mellin Transform is closely related to Fourier Transform, and is often used in number theory, mathematical statistics, the theory of asymptotic expansions and Fourier Transform was first introduce to solve PDEs and also has enormous applications in mathematical physics, engineering and applied sciences. Although the applications of these two integral transform are different from each other but combination of these two integral transform may be helpful in solving different problems which could not be solved by using these transforms separately.

In the present work, we have proved the Modulation and Parseval's Theorem of Two Dimensional Fourier-Mellin Transform.

KEYWORDS: Fourier Transform, Mellin Transform, Generalized function, Two dimensional Fourier-Mellin Transform.

INTRODUCTION

We know, the Modulation theorem of integral transform is a linear phase shift in time domain results in a frequency domain and the Parseval's theorem is the integral of the square of a function is equal to the integral of the square of its transform.

The Fourier transform (FT) is a linear, reversible transform with many more important properties. $F(s)$ is a Fourier transforms for any function $f(x)$, where x is a measure of the time domain and so s corresponds to the frequency-domain. It was first introduce to solve PDEs and also has enormous applications in mathematical physics, engineering [1]. By Bosi and Goldberg [2], FT is essential to understand how a signal behaves when it passes through filters, amplifiers and communications channels. It is applicable in image processing such as Transformation, representation, encoding, smoothing and sharpening images, It also uses in data analysis [3]. FT lies at the heart of signal processing including audio, speech, images, videos, seismic data, radio transmissions and also applicable in many modern technological advances including television, music CD's, DVD's, cell phones, movies, computer graphics, image processing, fingerprint analysis and storage, every mobile device such as net book, notebook, tablet, and phone have been built in high-speed cellular data connection [4].

Similarly in Mathematics, the Mellin transform is an integral transform and it may be regarded as the multiplicative version of the two-sided Laplace transform. It is closely related to the Fourier transform, the theory of the gamma function, allied special functions and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions. It also applicable in the analysis of water-wave problems such as the trapping of waves by submerged plates, this method seems to be wider applicability [5].

The Fourier-Mellin Transform (FMT) is used in computer, for recognition and differentiation of image of plant leaves due their translation, rotation and scale-invariance property [8]. The first investigator to use a version of Fourier-Mellin Transform was Brousil and Smith in 1967[6]. The next pioneer attempting at applying the FMT was investigated by Robbins and Huangs in 1972, they apply FMT for various optical distortion namely noise, in lenses [7]. Using [8], we can write the integral definition of Two Dimensional Fourier-Mellin Transform as-

The Two Dimensional Distributional Fourier-Mellin Transform with parameters s, u, p, v of $f(t, l, x, y)$ is given by-

$$FM\{f(t,l,x,y)\} = F(s,u,p,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\ e^{-i(st+ul)}\ x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

Also, we can write definition of the Distributional Two dimensional Fourier- Mellin Transform of Generalized $FM_{a,b,c,d,\alpha}$ function is-

$$FM\{f(t,l,x,y)\} = F(s,u,p,v) = \langle f(t,l,x,y), e^{-i(st+ul)}x^{p-1}y^{v-1}\rangle$$

Where, for each fixed$x(0 < x < \infty), y(0 < y < \infty), t(0 < t < \infty), l(0 < l < \infty)$. The R.H.S. of above has sense of an application of $f(t,l,x,y) \in FM^*_{a,b,c,d,\alpha}$ to $e^{-i(st+ul)}x^{p-1}y^{v-1} \in FM_{a,b,c,d,\alpha}$. $FM^*_{a,b,c,d,\alpha}$ is dual space of $FM_{a,b,c,d,\alpha}$. Here in the present paper, we generalized the two dimensional Fourier-Mellin Transform in the distributional sense.

From our previous work, the Inversion formula for the Distributional two Dimensional Fourier-Mellin Transform is defined as-

$$f(t,l,x,y)$$

$$= \lim_{r,\tau,r',\tau'\to\infty} \frac{1}{16\pi^4}\int_{-r}^{r}\int_{-\tau}^{\tau}\int_{-r'}^{r'}\int_{-\tau'}^{\tau'} F(s,u,p,v)\ e^{i(st+ul)}x^{-p}y^{-v}\,ds\,du\,dp\,dv$$

The outline of this paper-
Parseval's Theorem for the Distributional Two Dimensional Fourier-Mellin Transform is proved in Section 1. Section 2, gives the Modulation Theorem for the Distributional Two Dimensional Fourier-Mellin Transform. Lastly we conclude the present paper. The notations and Terminologies are given as per A.H. Zemanian[11].

PARSEVAL'S THEOREM

Theorem: If $FM\{f(t,l,x,y)\}(s,u,p,v) = F(s,u,p,v)$ and $FM\{g(t,l,x,y)\}(s,u,p,v) = G(s,u,p,v)$ then

i] $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,\overline{g(t,l,x,y)}\,dt\,dl\,dx\,dy$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} FM\{xyf(t,l,x,y)\}\,(s,u,p,v)\ \overline{G(s,u,p,v)}\,ds\,du\,dp\,dv$$

ii] $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}|f(t,l,x,y)|^2\ dt\,dl\,dx\,dy$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}|F(s,u,p,v)|^2\ ds\,du\,dp\,dv$$

Proof:-We have

$$FM\{f(t,l,x,y)\}(s,u,p,v) = F(s,u,p,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\ e^{-i(st+ul)}\ x^{p-1}y^{v-1}\ dt\ dl\ dx\ dy$$

By using Inversion formula for Two dimensional Fourier-Mellin Transform, We have-

$$f(t,l,x,y)$$

$$= \lim_{r,\tau,r',\tau'\to\infty} \frac{1}{16\pi^4}\int_{-r}^{r}\int_{-\tau}^{\tau}\int_{-r'}^{r'}\int_{-\tau'}^{\tau'} F(s,u,p,v)\ e^{i(st+ul)}x^{-p}y^{-v}\ ds\ du\ dp\ dv$$

$$f(t,l,x,y)$$

$$= \frac{1}{16\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(s,u,p,v)\ e^{i(st+ul)}x^{-p}y^{-v}\ ds\ du\ dp\ dv$$

$$f(t,l,x,y)$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} F(s,u,p,v)\ e^{i(st+ul)}x^{-p}y^{-v}\ ds\ du\ dp\ dv$$

Now it's conjugate is-

$$\overline{f(t,l,x,y)}$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \overline{F(s,u,p,v)}\ e^{-i(st+ul)}x^{p}y^{v}\ ds\ du\ dp\ dv$$

And

$$\overline{g(t,l,x,y)}$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \overline{G(s,u,p,v)}\ e^{-i(st+ul)}x^{p}y^{v}\ ds\ du\ dp\ dv$$

Consider,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,\overline{g(t,l,x,y)}\ dt\ dl\ dx\ dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\ dt\ dl\ dx\ dy$$

$$\left\{\frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \overline{G(s,u,p,v)}\ e^{-i(st+ul)}x^{p}y^{v}\ ds\ du\ dp\ dv\right\}$$

$$= \frac{1}{4\pi^4}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \overline{G(s,u,p,v)}\ ds\ du\ dp\ dv$$

$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} xy\ f(t,l,x,y)\ e^{-i(st+ul)}x^{p-1}y^{v-1}\ dt\ dl\ dx\ dy\right]\}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,\overline{g(t,l,x,y)}\ dt\ dl\ dx\ dy$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} FM[xy\ f(t,l,x,y)](s,u,p,v)\ \overline{G(s,u,p,v)}\ ds\ du\ dp\ dv$$

Putting-

$f(t,l,x,y) = g(t,l,x,y)$

$F(s,u,p,v) = G(s,u,p,v)$

$\overline{F(s,u,p,v)} = \overline{G(s,u,p,v)}$

By using the above result we have-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,\overline{f(t,l,x,y)}\,dt\,dl\,dx\,dy$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} F(s,u,p,v)\,\overline{F(s,u,p,v)}\,ds\,du\,dp\,dv$$

$$\int_{0}^{\infty}\int_{0}^{\infty}|f(t,l,x,y)|^2\;\,dt\,dl\,dx\,dy$$

$$= \frac{1}{4\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}|F(s,u,p,v)|^2\;\,ds\,du\,dp\,dv$$

Hence Proved.

MODULATION THEOREM

i] Prove that-

$$FM\{f(t,l,x,y)\cos(at+bl+cx+dy)\}(s,u,p,v)$$

$$= \frac{1}{2}\{FM[e^{i(cx+dy)}f(t,l,x,y)](s-a,u-b,p,v) + FM[e^{-i(cx+dy)}\,f(t,l,x,y)](s+a,u+b,p,v)\}$$

Proof:- We have

$$FM\{f(t,l,x,y)\cos(at+bl+cx+dy)\}(s,u,p,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\cos(at+bl+cx+dy)\;\,e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\left[\frac{e^{i(at+bl+cx+dy)}+e^{-i(at+bl+cx+dy)}}{2}\right]\,e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$= \frac{1}{2}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,e^{i(at+bl+cx+dy)}e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,e^{-i(at+bl+cx+dy)}\,e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy\}$$

$$= \frac{1}{2}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,e^{i(cx+dy)}e^{-i[(s-a)t+(u-b)l]}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\,e^{-i(cx+dy)}\,e^{-i[(s+a)t+(u+b)l]}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy\}$$

$$FM\{f(t,l,x,y)\cos(at+bl+cx+dy)\}(s,u,p,v)$$

$$= \frac{1}{2}\{FM[e^{i(cx+dy)}f(t,l,x,y)](s-a,u-b,p,v) + FM[e^{-i(cx+dy)}\,f(t,l,x,y)](s+a,u+b,p,v)\}$$

ii] Prove that-

$FM\{f(t,l,x,y)\sin(at+bl+cx+dy)\}(s,u,p,v)$

$$=\frac{1}{2}\{FM[e^{i(cx+dy)}f(t,l,x,y)](s-a,u-b,p,v)-FM[e^{-i(cx+dy)}f(t,l,x,y)](s+a,u+b,p,v)\}$$

Proof:- We have

$FM\{f(t,l,x,y)\sin(at+bl+cx+dy)\}(s,u,p,v)$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\sin(at+bl+cx+dy)\;e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\left[\frac{e^{i(at+bl+cx+dy)}-e^{-i(at+bl+cx+dy)}}{2i}\right]e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$=\frac{1}{2i}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\,e^{i(at+bl+cx+dy)}e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\,e^{-i(at+bl+cx+dy)}\,e^{-i(st+ul)}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy\}$$

$$=\frac{1}{2i}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\,e^{i(cx+dy)}e^{-i[(s-a)t+(u-b)l]}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(t,l,x,y)\,e^{-i(cx+dy)}\,e^{-i[(s+a)t+(u+b)l]}\,x^{p-1}y^{v-1}\,dt\,dl\,dx\,dy\}$$

$FM\{f(t,l,x,y)\sin(at+bl+cx+dy)\}(s,u,p,v)$

$$=\frac{1}{2i}\{FM[e^{i(cx+dy)}f(t,l,x,y)](s-a,u-b,p,v)-FM[e^{-i(cx+dy)}f(t,l,x,y)](s+a,u+b,p,v)\}$$

Hence Proved.

CONCLUSION

In the present work, Modulation and Parseval's Theorem for the Two dimensional Fourier-Mellin Transform are proved.

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